Approximate Methods: The Quartic Oscillartor in Momentum Space

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For unit mass the quartic oscillator has the following energy operator in atomic units in coordinate space.

$$H = -\frac{1}{2}\cdot\frac{d^2}{dx^2}\blacksquare + x^4\cdot\blacksquare$$

Suggested trial wave function: $\Psi(x,\beta) := \left(\frac{2\cdot\beta}{\pi}\right)^{\frac{1}{4}}\cdot\exp\left(-\beta\cdot x^2\right)$

Demonstrate that the wave function is normalized. $\int_{-\infty}^{\infty}\Psi(x,\beta)^2\,dx$ assume, $\beta > 0 \;\rightarrow 1$

Fourier transform the coordinate wave function into the momentum representation.

$$\Phi(p,\beta) := \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{-\infty}^{\infty}\exp(-i\cdot p\cdot x)\cdot\Psi(x,\beta)\,dx \;\Bigg|\begin{matrix}\text{assume}, \beta > 1\\ \text{simplify}\end{matrix} \rightarrow \frac{1}{2}\cdot\frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{4}}}\cdot\frac{e^{\frac{-1}{4}\cdot\frac{p^2}{\beta}}}{\beta^{\frac{1}{4}}}$$

Demonstrate that the momentum wave function is normalized. $\int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\Phi(p,\beta)\,dp$ assume, $\beta > 0 \;\rightarrow 1$

The quartic oscillator energy operator in momentum space: $H = \frac{p^2}{2}\cdot\blacksquare + \frac{d^4}{dp^4}\blacksquare$

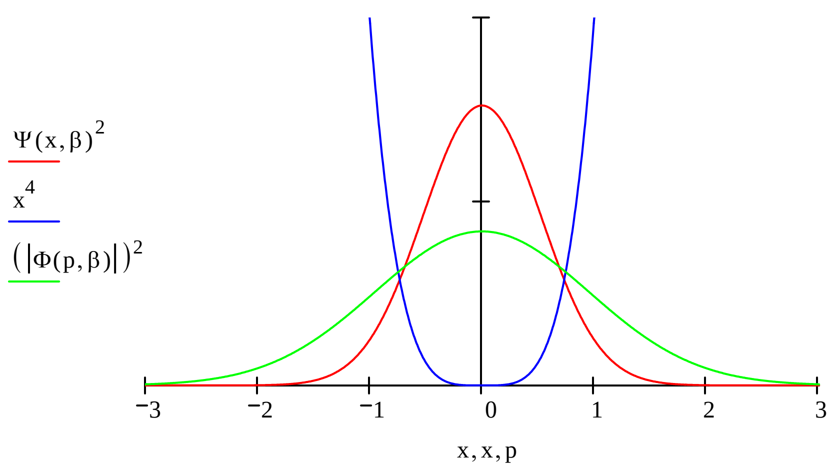
Evaluate the variational energy integral.

$$E(\beta) := \int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\frac{p^2}{2}\cdot\Phi(p,\beta)\,dp + \int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\frac{d^4}{dp^4}\Phi(p,\beta)\,dp \;\Bigg|\begin{matrix}\text{assume}, \beta > 0\\ \text{simplify}\end{matrix} \rightarrow \frac{1}{16}\cdot\frac{8\cdot\beta^3 + 3}{\beta^2}$$

Minimize the energy with respect to the variational parameter β and report its optimum value and the ground-state energy.

$\beta := 1$ $\quad\beta := \text{Minimize}(E,\beta)$ $\quad\beta = 0.90856$ $\quad E(\beta) = 0.68142$

Plot the coordinate and momentum wave functions and the potential energy on the same graph.



These results demonstrate the uncertainty principle. For the harmonic potential, $x^2/2$, the coordinate and momentum wave functions are identical. Compared to the harmonic potential the quartic potential, x^4, constrains the spatial wave function leading to less uncertainty in position. The uncertainty principle, therefore, requires an increase in the momentum uncertainty. This is clearly revealed in graph above.