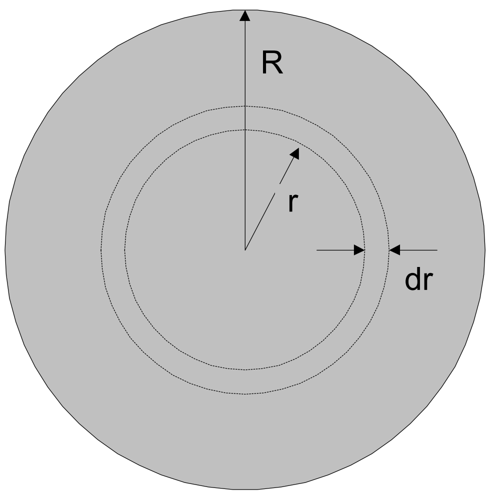
Physics 191
Calculating the moment of inertia (rotational inertia) of rigid bodies: A few worked examples

Example: Calculate the moment of inertia of a cylinder of mass *M* and radius *R* about an axis through the center of the cylinder.

The definition of the moment of inertia is $I = \int r^2 dm$, where we think about the integral as a sum over all of the infinitesimal masses *dm* that make up the object. In all of these calculations, the trick is to choose *dm* so that it each *dm* is at a well-defined distance from the axis of rotation—that is, at a well-defined value of *r*. Here, we will consider a cylinder divided up into *small cylindrical rings* of width *dr* and radius *r*, as shown in the drawing. For each ring, $dI = r^2 dm$, since *all* the mass of the ring is at a distance *r* from the axis of rotation (See Table 10-2, p. 253, item (a).)



First, we must find a way to write *dm* for the infinitesimal ring shown in the drawing. Begin by finding a relationship between the mass and the area of the cylinder; apparently

$$dm = \frac{M}{\pi R^2} dA_{area}$$

The reasoning here is similar to that used in the derivation of the moment of inertia of a thin bar: multiply the mass per unit area of the circle by the infinitesimal area *dA*. Thus, *dm* is just the mass per unit area of the cylinder multiplied by *dA*.

But the area of the ring is given by $dA = 2\pi r\, dr$ (that is, the circumference of the ring multiplied by the width *dr*). We find the moment of inertia by adding the contributions of all of the rings to the moment of inertia *I*.

$$I = \int_0^R r^2 dm = \int_0^R r^2 \frac{M}{\pi R^2} dA$$

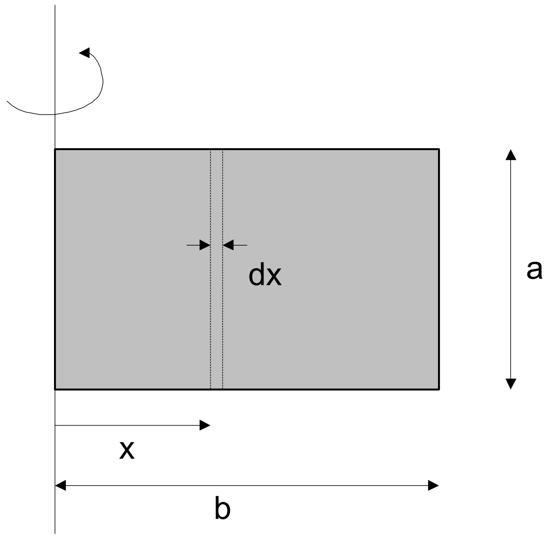
$$= \int_0^R r^2 \left(\frac{M}{\pi R^2} 2\pi r\, dr \right)$$

$$= \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \frac{r^4}{4} \Bigg|_0^R$$

$$= \frac{1}{2} M \frac{R^4}{R^2} = \frac{1}{2} MR^2$$

This result agrees with the table in the text (item c).

Example: Find the moment of inertia of a rectangular slab about an axis along one end.



Consider the moment of inertia of the small infinitesimal strip of length a and width dx, shown in the drawing. The moment of inertia of this strip is clearly

$$x^2 dm = x^2 \frac{M}{A}(a\ dx) = x^2 \frac{M}{ab}(a\ dx)$$

where M is the mass of the slab, A is the area of the slab, and the other quantities are defined in the drawing. Note that we are using the same trick that we used in the last example to find dm in terms of dx—that is, the area of the strip is $a\ dx$, and hence the mass dm of the strip is $\frac{M}{A}(a\ dx)$.

We find the moment of inertia of the slab by adding up the contributions of all of the strips:

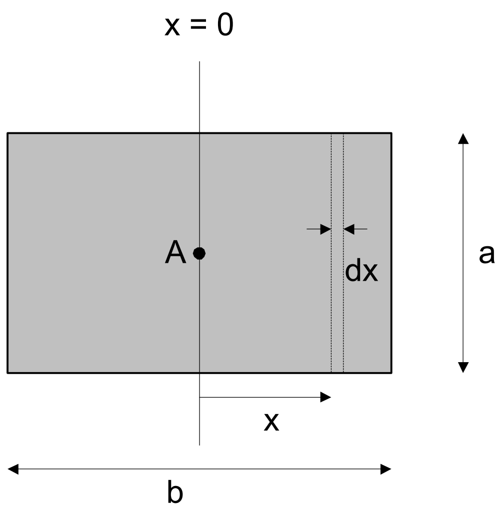
$$I = \int x^2 dm = \int_0^b x^2 \frac{M}{ab} a\ dx$$

$$= \frac{M}{b}\int_0^b x^2 dx = \frac{M}{b}\frac{x^3}{3}\Big|_0^b$$

$$= \frac{1}{3} Mb^3$$

This example is not included in the table in the book.

Example: Find the moment of inertia of a flat slab about an axis perpendicular to the slab and through its center of mass. (Note that the axis of rotation is perpendicular to the paper in this case.)



This example is a little trickier. We start by dividing the slab into infinitesimal strips. But this time, we consider each of these strips to be a long thin rod. The moment of inertia of a rod of length *a* about an axis perpendicular to the rod and through its center (that is, through its center of mass) is

$$dI_{cm} = \frac{1}{12}a^2 dm$$

Consider the moment of inertia *dI* of this same point about an axis through the center of the slab: the parallel axis theorem states that

$$I = I_{cm} + Md^2$$

As applied to this case, the moment of inertia *dI* of the infinitesimal strip about an axis through the center of the slab (that is, about point A in the drawing) is

$$\begin{aligned} dI &= dI_{cm} + dm\, x^2 \\ &= \frac{1}{12}a^2 dm + x^2 dm \\ &= \left(\frac{a^2}{12} + x^2\right) dm \end{aligned}$$

As in the previous example, we can write

$$dm = \frac{M}{A}(a\,dx) = \frac{M}{ab}(a\,dx) = \frac{M}{b}dx$$

As before, we find the moment of inertia of the slap by adding the moments of inertia of the disk:

$$I = \int_{-b/2}^{b/2} \left(\frac{a^2}{12} + x^2\right) \frac{M}{b}dx = \frac{M}{b}\left(\frac{a^2}{12}x + \frac{x^3}{3}\right)\Bigg|_{-b/2}^{b/2}$$

$$= \frac{M}{b}\left(\frac{a^2}{12}b + \frac{b^3}{12}\right) = \frac{1}{12}M(a^2 + b^2)$$

which is the result given for item (i) in the table on page 253. Note that I have skipped a few steps in the algebra!