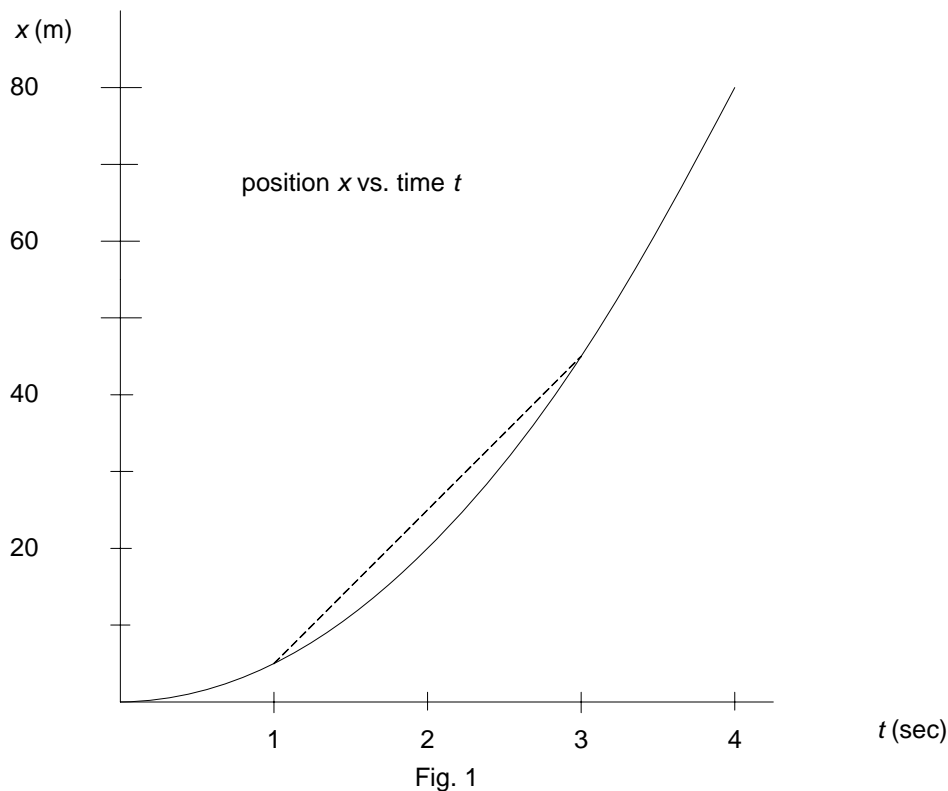


INSTANTANEOUS VELOCITY

I claim that that if acceleration is constant, then the velocity is a linear function of time and the position a quadratic function of time. We want to investigate those claims, and at the same time, work out what we mean by “instantaneous velocity.”

Assume that the relation between distance and time is given by $x(t) = 5t^2$. Suppose we want to find the (instantaneous) velocity at $t = 1$ sec. We start with a graph.



Let's find the average velocity. In general,

$$v_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \quad (1)$$

Let $t_1 = 1$ sec and $t_2 = 3$ sec. Then the average velocity is just the slope of the line (dashed line in drawing) connecting these two points:

$$v_{av} = \frac{5t_2^2 - 5t_1^2}{t_2 - t_1} = \frac{5 \cdot 9 - 5 \cdot 1}{3 - 1} = 20 \text{ m/s}$$

To find the instantaneous velocity, let's make this time increment smaller and smaller. First, let $t_1 = 1$ sec and $t_2 = 2$ sec.

$$v_{av} = \frac{5t_2^2 - 5t_1^2}{t_2 - t_1} = \frac{5 \cdot 4 - 5 \cdot 1}{2 - 1} = 15 \text{ m/s}$$

Now let's try $t_2 = 1.5$ sec. Then $v_{av} = 1 \frac{5 \cdot (1.5)^2 - 5 \cdot 1}{1.5 - 1} = 12.5 \text{ m/s}$. And so on.

Exercise: Show that for $t_2 = 1.1, 1.01, \dots$ sec, $v_{av} = 10.5, 10.05, \dots$ m/s.

The average velocity seems to be getting closer and closer to 10 m/s, as we t_2 gets closer and closer to t_1 . Will this pattern hold?

Let's investigate the question algebraically. For our function, the average velocity will be

$$v_{av} = \frac{5t_2^2 - 5t_1^2}{t_2 - t_1}.$$

Now, let's define a small time increment Δt that we can make as small as we want. We can now write $t_2 = t_1 + \Delta t$, and our expression for average velocity becomes

$$v_{av} = \frac{5(t_1 + \Delta t)^2 - 5t_1^2}{(t_1 + \Delta t) - t_1}$$

With a little algebra, this equation becomes

$$\begin{aligned} v_{av} &= \frac{5(t_1 + \Delta t)^2 - 5t_1^2}{(t_1 + \Delta t) - t_1} = \frac{(5t_1^2 + 10t_1\Delta t + 5\Delta t^2) - 5t_1^2}{\Delta t} \\ &= \frac{10t_1\Delta t + 5\Delta t^2}{\Delta t} \\ &= 10t_1 + 5\Delta t \end{aligned}$$

Exercise: Show that the appropriate choice of Δt gives the same values for the average velocity we found above.

If we now let Δt shrink to zero, the average velocity becomes the instantaneous velocity at the time t_1 :

$$v = 10t$$

where for convenience I have dropped the subscript. In mathematical language, we have taken a limit, and in the process have taken a derivative—the slope of the line tangent to the curve—at any arbitrary time. The instantaneous velocity is thus the derivative of the position.

We have also shown that if the position is a quadratic function of time, then the velocity is linear in the time—as claimed. **Exercise:** Can you go on to show that the acceleration is constant in this example?

We have also raised a whole host of mathematical questions: Have I inadvertently divided by zero above, for example? More generally, does this limit always exist? These questions and others we may safely leave to your mathematics courses!

A non-calculus approach to instantaneous velocity

Galileo got these same results, decades before Newton and Leibnitz invented the calculus. Here is (more or less) how he did it. We first notice that if the acceleration is constant, a graph of velocity vs. time will give a straight line. To understand this point, recall that the average acceleration is just the slope of a graph of velocity vs. time; so if the graph of velocity vs. time is a straight line, the slope—the average acceleration—always has the same value.

So now we know that if acceleration is constant, the velocity is a linear function of time. If we can show that position is a quadratic, we will be done! We will need the **mean speed theorem**—this theorem was known mathematically in the Middle Ages, and Galileo applied it to motion. It states the following:

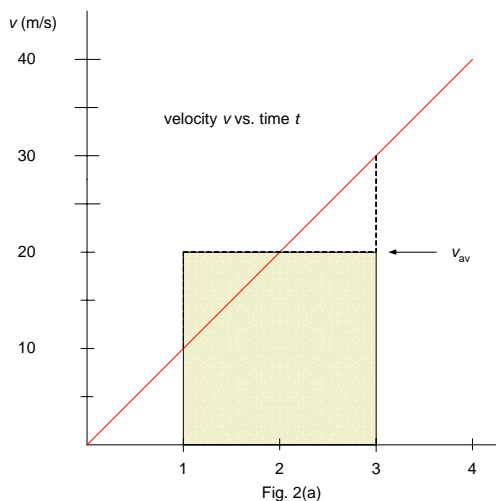


Fig. 2(a)

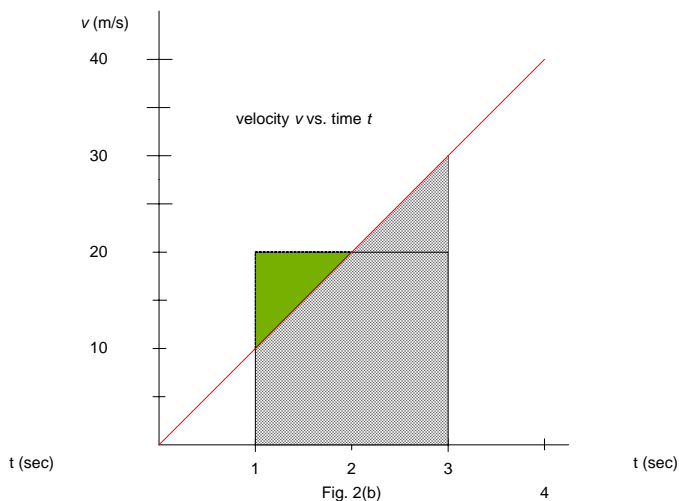


Fig. 2(b)

Mean Speed Theorem: If acceleration is constant, the distance traveled in a time interval is the same as the distance the object would travel at the constant “mean speed”—the average of the initial and final velocities for the same time interval.¹ (**Note:** YF talks about this theorem, without giving it a name, on page 48—see especially Equation 2-10.) To see why this theorem is true, consider the graphs of velocity vs. time shown above:

- In the first graph (Fig 2(a)), the distance traveled by an object moving at the mean speed is the mean speed multiplied by the time interval:

$$\Delta x = v_{\text{mean}} \Delta t$$

where $v_{\text{mean}} = (v_{\text{initial}} + v_{\text{final}}) / 2$. This result should be evident from the figure.

¹ In other words, the average velocity defined in Eq. (1) is equal to the mean speed—the average of the initial and final velocities—ONLY if the acceleration is constant. Note from Fig. (2(a)) that this mean speed is also the instantaneous speed at the middle of the time interval.

- Notice that this distance, Δx , is just the area of the rectangle shown in the graph.
- Now consider the second graph (Fig. 2(b)).
 - The areas of the green and gray triangles in the second graph are equal.
 - Intuitively, the distance traveled by the uniformly accelerated object is likewise the area under the line (gray area in Fig. 2(b)).
 - Therefore, the shaded areas in Fig. 2(a) and Fig. 2(b) are the same: The uniformly accelerated object travels the same distance that it would if it were moving at the (constant) mean velocity. Or in other words, the mean velocity is just the average velocity as defined in Eq. (1). **We have just proven the mean speed theorem.**
- The rest is straightforward: If we call $v_{\text{initial}} = v_1$ and $v_{\text{final}} = v_2$, then the distance traveled is

$$\Delta x = v_{av} \Delta t = \frac{1}{2}(v_1 + v_2)\Delta t \text{ since, as we have just shown, } v_{av} = \frac{1}{2}(v_1 + v_2).$$

As usual, $\Delta t = t_2 - t_1$. But from our definition of average acceleration,

$$a = \frac{v_2 - v_1}{t_2 - t_1}.$$

After a line or two of algebra, this equation becomes

$$v_2 = v_1 + a(t_2 - t_1) = v_1 + a\Delta t$$

and therefore

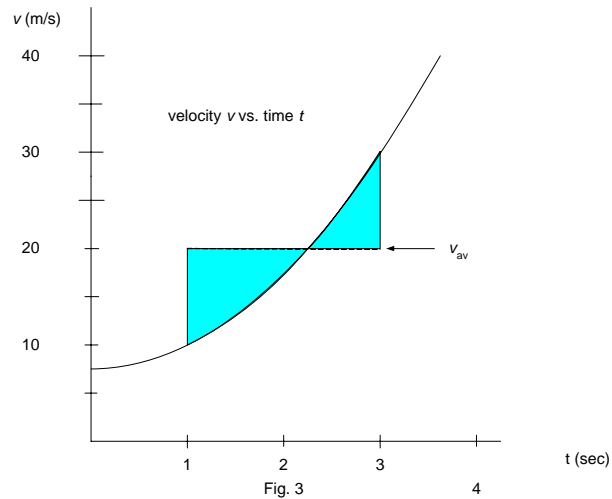
$$\begin{aligned} \Delta x &= \frac{1}{2}(v_1 + v_2)\Delta t = \frac{1}{2}(v_1 + [v_1 + a\Delta t])\Delta t \\ &= v_1\Delta t + \frac{1}{2}a(\Delta t)^2 \end{aligned}$$

That result is what we expect. It's a little simpler if we let $\Delta t = t$ (that is, we let our clock start at $t_1 = 0$ and let $t_2 = t$). In that case we have

$$\Delta x = v_1 t + \frac{1}{2} a t^2.$$

Thus, without using calculus, we have shown that if acceleration is constant, velocity is a linear function of time, and displacement a quadratic function of time! Compare this result to YF Eq. (2.12)

- Finally, consider Fig. 3, shown below. The acceleration is no longer constant, so the



velocity is no longer linear. Notice that the mean velocity is the same as before: the average of the velocity at $t = 1$ s and $t = 3$ s. But the areas of the two “triangular” segments are no longer equal: The mean speed theorem does *not* work for this case—it works *only* when the acceleration is constant!