Work-Energy Theorem
Three Dimensions, non-constant Forces

(See Feynman, Lectures on Physics, Vol. I, Chapter 13)

Recall that the Work-Energy theorem states that

Change in kinetic energy = Work done, or

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = W$$

We have proven this theorem for the cases of constant forces (using the methods of chapter 2). YF give a proof in Chapter 6 for non-constant forces, but only in one dimension. Here we remove all restrictions and consider non-constant forces in two or three dimensions. The proof works as follows: Begin by considering

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \text{rate of change of kinetic energy}$$

But we can write the kinetic energy in terms of the scalar product, since $v^2 = \vec{v} \circ \vec{v}$. Hence

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= \frac{1}{2}m\frac{d}{dt}\vec{v} \circ \vec{v} \\ &= \frac{1}{2}m\left(\frac{d\vec{v}}{dt} \circ \vec{v} + \vec{v} \circ \frac{d\vec{v}}{dt}\right) \\ &= m\frac{d\vec{v}}{dt} \circ \vec{v} \end{aligned}$$

where we have used standard rules of differentiation.

But by Newton II, $m\dfrac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$; hence, substituting in the last equation, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \vec{F} \circ \vec{v}$$

This result is interesting in itself, since it gives us an ***alternative derivation of the equation for power*** (See YF Chapter 6, Section 4, Eq. 6.19). But it also leads us back to the Work-Energy Theorem. We know that the velocity is $\vec{v} = \dfrac{d\vec{r}}{dt}$ where $\vec{r}$ is a displacement vector. Hence

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \vec{F} \circ \frac{d\vec{r}}{dt}$$

If we integrate this equation with respect to the time *t* we obtain

$$d\left(\frac{1}{2}mv^2\right) = \vec{F} \circ d\vec{r}$$

But this equation is simply a differential form of the Work-Energy Theorem. We need only integrate this equation (this step may stretch your mathematics a bit):

$$\int_{v_i}^{v_f} d\left(\frac{1}{2}mv^2\right) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \circ d\vec{r} = W \quad \text{or}$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$$

At this point, we have proven the Work-Energy Theorem for the most general case!