

## Summary of Rotational Motion

First, recall essentials of forces and Newton's second law:

$$\text{momentum: } \vec{p} = m\vec{v}$$

$$\text{Newton II: } \vec{F} = \frac{d\vec{p}}{dt}$$

For the special case when the mass is constant, we found that Newton II can be expressed

$$\text{usual form of Newton II: } \vec{F} = m\vec{a}$$

Furthermore, we discovered (YF Chapter 8) that

- If no *external* forces act on a collection of masses, we can show that internal forces cancel in pairs (Newton III), and hence the total momentum is constant; and
- if external forces do act, the collection of masses can be treated as if the net force acts on the center of mass—it is as if all the mass were concentrated at that one point.

These ideas carry over into an analysis of rotational motion. In particular, we introduced

**torque:**  $\vec{\tau} = \vec{r} \times \vec{F}$  where  $\vec{F}$  is a force and  $\vec{r}$  is a displacement vector from the point about which the torque is calculated (typically but not necessarily an axis of rotation) to the point where the force is applied. Intuitively, we saw that a torque gives rise to rotational motion.

**angular momentum:**  $\vec{L} = \vec{r} \times \vec{p} = r \times m\vec{v}$  where  $\vec{p}$  is the momentum, and  $\vec{r}$  a displacement vector defined in the same way as above.

These definitions of torque and angular momentum led us to a “rotational form” of Newton's second law:

$$\vec{\tau} = \frac{d\vec{L}}{dt}.$$

Moreover, this form implies the *conservation of angular momentum*: If the total external torque acting on a collection of masses or a rigid body is zero, then the angular momentum is a constant. This result allows one to explain a number of curious rotational phenomena, including the demonstrations we did in class. Intuitively, the conservation of angular momentum says that objects tend to keep rotating, and that a torque is needed to change the rotational state of a system of masses.

These definitions of torque and angular momentum can be applied to just about any configuration of masses, but they are particularly useful when applied to rotational motion. In particular, we have considered the following special cases:

**Case 1:** Rotation about a *fixed* axis (YF Chapter 10) for torque and angular momentum. For this case, the above definitions reduce to

$$\vec{\tau} = I\vec{\alpha}$$

$$\vec{L} = I\vec{\omega}$$

where  $\omega$  and  $\alpha$  are respectively the angular velocity and angular acceleration. Recall that we obtained these results by considering the motion of a mass point in a circle, and applying the general

definitions of torque and angular momentum. We then applied these ideas to specific problems by drawing force diagrams, and writing the appropriate forms of Newton II for forces and torques.

We also found that the kinetic energy of a “rigid body” rotating about a fixed axis (Chapter 9) to be

$$KE = \frac{1}{2} I \omega^2$$

and used this form apply energy conservation methods to problems involving rotation.

**Case 2:** We consider rotation about an axis that is not fixed, but that does always remain parallel to itself—for example, a cylinder rolling down an inclined plane (YF Chapter 10). We considered the case of an object of radius  $R$  rolling *without slipping*, and showed that the velocity and acceleration of the center of mass are given by

$$v_{\text{cm}} = \omega R \quad \text{and} \quad a_{\text{cm}} = \alpha R$$

We obtained these results by considering the diagrams in the first few pages of YF Chapter 10. Notice that this same relations apply to a falling yo-yo—the argument is much the same. It followed that the total kinetic energy of such an object could be written

$$KE = \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2 ;$$

that is, the kinetic energy of such an object is the sum of the translational kinetic energy of the center of mass, and the rotational kinetic energy about the center of mass. This result turns out to be a very general one.

We also stated (but did not prove) that as long as the axis of rotation stays parallel to itself, the torque about the center of mass can always be written

$$\tau_{\text{cm}} = I \alpha_{\text{cm}}$$

*even if the rotating object is accelerating!!* This conclusion is surprising—we do not expect Newton’s laws to work in accelerating coordinate systems. And indeed, this result is not true for points other than the center of mass.

We now have an approach to analyzing the linear and angular accelerations of rigid bodies, even when the center of mass is accelerating, as long as the axis of rotation remains parallel to itself:

- Use the principle from chapter 8 that the as far as translational motion is concerned, the total force can be considered as acting on the center of mass; hence  $F = m a_{\text{cm}}$ .
- Use  $\tau_{\text{cm}} = I \alpha_{\text{cm}}$  to calculate the torque even when the center of mass is accelerating. We used this approach to analyze such problems as the falling yo-yo and a rigid body rolling without slipping down an inclined plane.

Be sure you understand how the details of this analysis work—it’s similar to our analysis of forces and accelerations in chapters 4 and 5. Rely on the worked examples and the homework problems.

Remember also that we sometimes approached such problems through the conservation of energy. In this case, we noted that the change in the potential energy of a rigid body could be calculated on the assumption that the entire mass is concentrated at the center of mass.