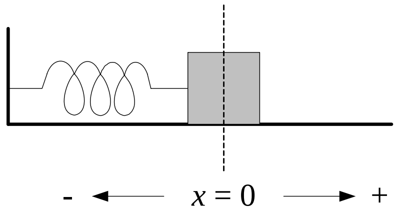
Simple Harmonic Motion

A mass attached to a spring oscillates with a period that depends on the mass and the spring constant. Here, I want to analyze this motion, using Newton's second law, in more depth than the text.

We have already seen that springs (at least, ideal ones) obey Hooke's law. If x is the displacement of a mass from equilibrium, Hooke's law says that



$$F = -kx \tag{1}$$

Here k is the spring constant, and we have arbitrarily taken positive to the right. The minus sign in the equation tells us that F is a *restoring* force—one that always pushes the mass back towards $x = 0$. For example, if x is positive (to the right), F is negative (to the left).

Now we apply Newton's second law:

$$F = -kx = ma \quad \text{or}$$

$$a = -\frac{k}{m}x \tag{2}$$

Notice that the acceleration is *not* constant—it depends on the displacement x.

We seek an equation for $x(t)$ that describes the position of the mass as a function of time. To find it, we need some calculus. We know that in one dimension, the acceleration is just the second derivative of the displacement:

$$a = \frac{d^2x}{dt^2}$$

We substitute this result in Eq. (2) to obtain

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{3}$$

Eq. (3) is a simple example of a "differential equation." We solve it for x as a function of time in what may seem like an unusual way: First we guess at a solution:

Guess: $$x = A\cos(\omega t + \phi) \tag{4}$$

We call A the amplitude (the maximum value of x), ω the angular frequency in radians/sec, and φ the phase constant that specifies the initial (t = 0) position and velocity of the mass.

We next see if this guess works—it must be mathematically consistent; it cannot lead to nonsensical physical results; and there should be some way to test it experimentally.

Thus, our next step is to take two derivatives of Eq. (4) and substitute for the left side Eq. (3):

$$x = A\cos(\omega t + \phi)$$
$$\frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$
$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

We substitute this result into Eq. (3) to obtain:

$$-\omega^2 A\cos(\omega t + \phi) = -\frac{k}{m}x$$

Finally we substitute our guess (Eq. (4)) for *x* on the right-hand side to obtain

$$-\omega^2 A\cos(\omega t + \phi) = -\frac{k}{m}A\cos(\omega t + \phi)$$

A little algebra leads us to the result

$$A\cos(\omega t + \phi)\left[-\omega^2 + \frac{k}{m}\right] = 0$$

This equation can be true for all times only if the term in the square bracket is zero:

$$-\omega^2 + \frac{k}{m} = 0 \quad \text{or}$$
$$\omega = \sqrt{\frac{k}{m}} \qquad (5)$$

But this result is perfectly consistent! Hence it follows that a mass-spring system obeying Hooke's law—the "simple harmonic oscillator"—must, according to Newton's second law, obey the equation

$$x = A\cos(\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}.$$

Exercise: To persuade yourself that this method actually works, try a second guess: $x = \frac{1}{2}gt^2$, our result for a freely falling mass experiencing constant acceleration. We know that this guess *cannot* be right, since for Hooke's law, the acceleration cannot be constant. Try this guess anyway, just as we have done above, and show that it leads to an absurd result.