Elastic collisions
One dimension
One mass initially at rest

The equations for conservation of momentum and energy are

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{and}$$

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

We solve the first equation for v_{2f}:

$$v_{2f} = \frac{m_1 v_{1i} - m_1 v_{1f}}{m_2}$$

and substitute into the second:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 \left(\frac{m_1 v_{1i} - m_1 v_{1f}}{m_2} \right)^2$$

Our task now is to simplify and solve for v_{1f}. To start,

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + \frac{m_2 m_1^2}{m_2^2} \left(v_{1i} - v_{1f} \right)^2 .$$

Simplifying further

$$v_{1i}^2 = v_{1f}^2 + \frac{m_1}{m_2} \left(v_{1i} - v_{1f} \right)^2 .$$

Now, square the term in parentheses, and move everything to the right-hand side:

$$0 = -v_{1i}^2 + v_{1f}^2 + \frac{m_1}{m_2} \left(v_{1i} - v_{1f} \right)^2 .$$

Note that the first two terms can be written as

$$0 = \left(v_{1f} - v_{1i} \right)\left(v_{1f} + v_{1i} \right) + \frac{m_1}{m_2} \left(v_{1i} - v_{1f} \right)^2$$

or, rearranging signs

$$0 = -\left(v_{1i} - v_{1f} \right)\left(v_{1f} + v_{1i} \right) + \frac{m_1}{m_2} \left(v_{1i} - v_{1f} \right)^2 .$$

We now divide both sides by $\left(v_{1i} - v_{1f} \right)$ to obtain

$$0 = -\left(v_{1f} + v_{1i} \right) + \frac{m_1}{m_2} \left(v_{1i} - v_{1f} \right) \quad \text{or}$$

$$0 = -v_{1f} - v_{1i} + \frac{m_1}{m_2} v_{1i} - \frac{m_1}{m_2} v_{1f}$$

Collecting terms,

$$0 = -v_{1f}\left(1+\frac{m_1}{m_2}\right)+v_{1i}\left(-1+\frac{m_1}{m_2}\right).$$

We move the first term to the left-hand side

$$v_{1f}\left(1+\frac{m_1}{m_2}\right)=v_{1i}\left(-1+\frac{m_1}{m_2}\right)$$

and solve for v_{1f}

$$v_{1f} = \frac{-1+\frac{m_1}{m_2}}{1+\frac{m_1}{m_2}} = \frac{\left(-1+\frac{m_1}{m_2}\right)m_2}{\left(1+\frac{m_1}{m_2}\right)m_2}.$$

Where in the last step we have multiplied top and bottom by m_2. Clearing the fractions, we obtain

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$

the result we are looking for! To find v_{2f} we substitute the above equation into our previous result:

$$\begin{aligned} v_{2f} &= \frac{m_1 v_{1i} - m_1 v_{1f}}{m_2} = \frac{m_1}{m_2}\left(v_{1i} - v_{1f}\right) \\ &= \frac{m_1}{m_2}\left(v_{1i} - \frac{m_1 - m_2}{m_1 + m_2} v_{1i}\right) \end{aligned}$$

I leave it as an exercise to put the terms in the parentheses under a common denominator and show

$$v_{2f} = \frac{m_1}{m_2}\left(\frac{2m_2}{m_1 + m_2} v_{1i}\right) = \frac{2m_1}{m_1 + m_2} v_{1i}$$

which is the result we want. We have now shown that for an elastic collision in 1D,

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad \text{and} \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

Remember that these results apply *only* for elastic collisions—collision in which both momentum and kinetic energy are conserved.