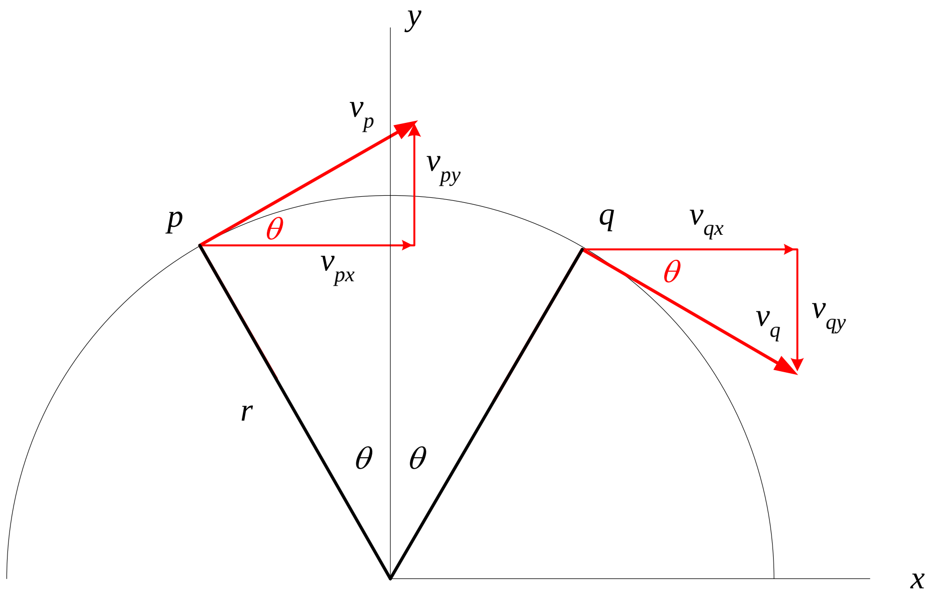
Kinematics of Uniform Circular Motion

We want to show that the centripetal acceleration for a particle in uniform circular motion is

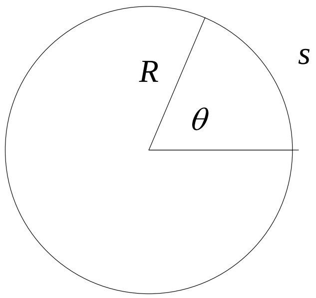
$$a = \frac{v^2}{r},$$

and directed towards the center of the circle. Here v is the (constant) speed and r the radius of the circle. In this derivation, we will not need any calculus; we will, however, need a fair amount of algebra and geometry.



Strategy: In the diagram shown above, assume that the particle moves from p to q in a time Δt seconds, at a constant speed v. We will calculate the average acceleration, and then take the limit as $\Delta t \to 0$.

1. First, that as the particle moves from p to q, it moves a distance $\text{arc}(pq)$, where θ is measured in radians. Now recall the *definition* of angle: In the drawing below, if s is arc length and r is the radius of a circle, then by definition



$$\theta \equiv \frac{s}{R} \quad \text{or} \quad s = R\theta.$$

Hence in our case, in the first drawing, the length of the arc from p to q is

$$\text{arc}\,(pq) = r(2\theta).$$

Since the particle is moving at a constant speed v, we can find the time Δt it takes to move from p to q using the notion that distance = constant speed × time:

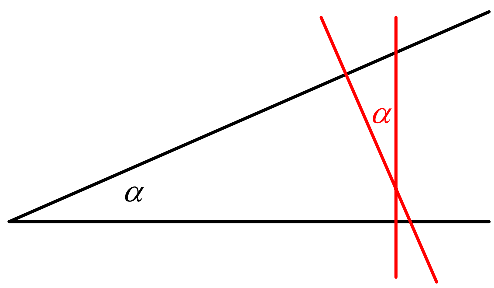
$$\Delta t = \frac{r(2\theta)}{v}$$

2. We now find the x and y components of the average acceleration as the particle moves from p to q. First the x component (the overhead bar means "average"):

$$\begin{aligned}\bar{a}_x &= \frac{v_{x\ \text{final}} - v_{x\ \text{initial}}}{\Delta \text{t}} = \frac{v_{qx} - v_{px}}{\Delta \text{t}} \\ &= \frac{v\cos\theta - v\cos\theta}{\Delta \text{t}} = 0\end{aligned}$$

   where in the second line I have written v_x in terms of the speed v and the angle θ. It's not hard to see from the diagram that the average x-component of the acceleration is zero.

   Note: Be sure you understand why the all the angles in the first drawing are the same—in terms of my drawing, why the angles shown in black are equal to those shown in red. It's an example of the ***mutual perpendiculars theorem*:**



   Suppose two lines intersect at an angle α.

   Draw lines perpendicular to the original lines (red in diagram)—then the two new lines also meet at an angle α.

3. Now the y component of the acceleration:

$$\begin{aligned}\bar{a}_y &= \frac{v_{y\ \text{final}} - v_{y\ \text{initial}}}{\Delta \text{t}} = \frac{v_{qy} - v_{py}}{\Delta \text{t}} \\ &= \frac{-v\sin\theta - v\sin\theta}{\Delta \text{t}} = \frac{-2v\sin\theta}{\Delta \text{t}} \\ &= -\frac{2v\sin\theta}{\left(\frac{2r\theta}{v}\right)} = -\frac{v^2}{r}\frac{\sin\theta}{\theta}\end{aligned}$$

   where in the third line I have substituted for Δt from above, and simplified. Notice the negative sign—it means that the average acceleration is *down*. **Note:** *Here is where we are showing that the acceleration is directed toward the center of the circle.*

4. Finally, we take the limit as $\Delta t \to 0$. Equivalently, we can let $\theta \to 0$. But now we can use a well-known theorem from calculus:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

We can find this result numerically: Set your calculator to radian mode and try calculating $\sin\theta / \theta$ for smaller and smaller angles. Those of you who have studied Taylor's series expansions will also know the result

$$\sin\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \ldots$$

from which our result follows immediately. Hence we know that *in the limit*, the acceleration is

$$a = \frac{v^2}{r}$$

and directed to the center of the circle: QED!